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Unifying the theory of Integration within normal-, Weyl- and antinormal-ordering of operators and the s–ordered operator expansion formula of density operators

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By introducing the s–parameterized generalized Wigner operator into phase-space quantum mechanics we invent the technique of integration within s–ordered product of operators (which considers normal ordered, antinormally ordered and Weyl ordered product of operators as its special cases). The s–ordered operator expansion (denoted by $\S\cdots\S$) formula of density operators is derived, which is

$$\rho=\frac{2}{1-s}\int\frac{d^2\beta}{\pi}\left\langle -\beta\right|\rho\left|\beta\right\rangle \S\exp\{\frac{2}{s-1}\left(s|\beta|^2-\beta^*a+\beta a^\dagger-a^\dagger a\right)\}\S,$$

The s–parameterized quantization scheme is thus completely established.

Keywords: s–parameterized generalized Wigner operator, technique of integration within s–ordered product of operators, s–ordered operator expansion formula, s–parameterized quantization scheme

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I. INTRODUCTION

The subject about operators and their classical correspondence has been a hot topic since the birth of quantum mechanics (QM) and now becomes a field named QM in phase space. Because Heisenberg's uncertainty principle prohibits the notion of a system being described by a point in phase space, only domains of minimum area $2\pi\hbar$ in phase space is allowed. Wigner [1] introduced a function whose marginal distribution gives probability of a particle in coordinate space or in momentum space, respectively. The Wigner distribution is related to operators' Weyl ordering (or Weyl quantization scheme) [2]. We notice that each phase space distribution is associated with a definite operator ordering for quantizing classical functions. For examples, P-representation (as a density operator ρ's classical correspondence) is actually ρ's antinormally ordered expansion in terms of the completeness of coherent state $\left|z\right\rangle=\exp[-\frac{|z|^2}{2}+za^\dagger]\left|0\right\rangle$ [3, 4],

$$\rho=\int\frac{d^2z}{\pi}P\left(z\right)\left|z\right\rangle\left\langle z\right| \tag{1}$$

because the coherent states compose a complete set $\int\frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right|=1$ [5]. The Wigner distribution function $W\left(p,x\right)$ of ρ, defined as $Tr\left[\rho\Delta\left(p,x\right)\right]$, is proportional to the classical Weyl correspondence $h\left(p,x\right)$ of ρ (ρ's Weyl ordered expansion), i.e.,

$$\rho=\iint_{-\infty}^{\infty}dpdx\Delta\left(p,x\right)h\left(p,x\right), \tag{2}$$

$$Tr\left[\rho\Delta\left(p,x\right)\right]=\left(2\pi\right)^{-1}h\left(p,x\right)=W\left(p,x\right). \tag{3}$$

since the Wigner operator $\Delta\left(p,x\right)$ is complete too, $\iint_{-\infty}^{\infty}dpdx\Delta\left(p,x\right)=1$. The original form of $\Delta\left(p,x\right)$ defined in the coordinate representation is [6]

$$\Delta\left(x,p\right)=\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{iup}\left|x+\frac{u}{2}\right\rangle\left\langle x-\frac{u}{2}\right|, \tag{4}$$

for the Wigner operator in the entangled state representation we refer to [7]. When $\rho = \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l$, $[X, P] = i$, $\hbar = 1$, according to Eqs. (3)-(4), the classical correspondence of $\left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l$ is

$$\begin{aligned}
&2\pi Tr\left[\left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l \Delta(x,p)\right] \\
&= \int_{-\infty}^{\infty} du e^{ipu} \left\langle x - \frac{u}{2}\right| \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \frac{m!}{l!\,(m-l)!} X^{m-l} P^r X^l \left| x + \frac{u}{2}\right\rangle \\
&= x^m \int_{-\infty}^{\infty} du e^{ipu} \left\langle x - \frac{u}{2}\right| P^r \left| x + \frac{u}{2}\right\rangle \\
&= x^m \int_{-\infty}^{\infty} du e^{ipu} \int_{-\infty}^{\infty} dp' e^{-ip'u} p'^r \\
&= x^m \int_{-\infty}^{\infty} dp' \delta(p - p') p'^r \\
&= x^m p^r,
\end{aligned} \tag{5}$$

this is the original definition of Weyl quantization scheme (quantizing classical coordinate and momentum quantity $x^m p^n$ as the corresponding operators) as [2]

$$x^m p^n \to \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l, \tag{6}$$

its right-hand side is in Weyl ordering, so we introduce the symbol $\vdots\,\vdots$ to characterize it [8], i.e.,

$$\left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l = \vdots \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l \vdots, \tag{7}$$

It is worth emphasizing that the order of operators X and P are permuted within the Weyl ordering symbol [8], a useful property which has been overlooked for a long time, Based on this fact a useful method called integration within Weyl ordered product of operators has been invented [8].

Therefore, from Eq. (6) and Eq. (7)

$$x^m p^r \to \vdots \left(\frac{1}{2}\right)^m \sum_{l=0}^{m} \binom{m}{l} X^{m-l} P^n X^l \vdots = \vdots X^m P^r \vdots. \tag{8}$$

Following Eq. (11) we have

$$\vdots X^m P^r \vdots = \int\int_{-\infty}^{\infty} dp dx \Delta(x,p)\, x^m p^r, \tag{9}$$

which implies $\Delta(x,p) = \vdots \delta(x - X)\, \delta(p - P) \vdots$, or $\Delta(\alpha) = \frac{1}{2} \vdots \delta\left(\alpha^* - a^\dagger\right) \delta(\alpha - a) \vdots$, $\alpha = (x + ip)/\sqrt{2}$, a delta operator-function form in Weyl ordering.

Having realized that each phase space distribution accompanies a definite operator ordering for quantizing classical functions, we may think of that each complete set of operators corresponds to an operator-ordering rule. In this work we shall introduce a complete set of operators characteristic of a s-parameter (the generalized Wigner operator) and then introduce a generalized quantization scheme with the s-parameter operator ordering. Historically, Cahill and Glauber [9] have introduced the s–parameterized quasiprobability distribution according to which the coherent state expectation of ρ, the Wigner function of ρ, and the P-representation of ρ respectively corresponds to three distinct values of s, i.e., $s = 1, 0, -1$. However, the s–parameterized quantization scheme associated with the s–parameterized quasiprobability distribution has not been completely established, as the fundamental problem of what is ρ's s-ordered operator expansion has not been touched yet. In another word, the problem of how to arrange any given operator as its s-ordered form has been unsolved, say for instance, no references has ever reported what is the s-ordered operator expansion of $exp\left(\lambda a^\dagger a\right)$? ($[a, a] = 1$) In this work we shall solve this important problem by introducing the technique of integration within s-ordering of operators, which in the cases of

$s = 1, 0, -1$, respectively goes to the technique of integration within normal-ordering, Weyl ordering and antinormal ordering of operators. In this way we can tackle these three techniques in a unified way. The work is arranged as follows: In Sec. 2 we introduce the explicit s–parameterized Wigner operator $\Delta_s(\alpha)$ and then in Sec. 3 we establish one-to-one mapping between operators and their s–parameterized classical correspondence after proving the relation $2\pi Tr\left[\Delta_{-s}(\alpha'^*, \alpha')\Delta_s(\alpha^*, \alpha)\right] = \delta(x'-x)(p'-p)$, where $\alpha = (x+ip)/\sqrt{2}$. In Sec. 4 we introduce the symbol $\S\cdots\S$ denoting s–ordering of operators and the technique of integration within s–ordered product of operators. In Sec. 5-6 we derive density operator's expansion formula in terms of s–ordered quantization scheme, such that the s-ordered expansion of $exp\left(\lambda a^\dagger a\right)$ is obtained. In this way we develop and enrich the theory of phase space quantum mechanics.

II. THE S–PARAMETERIZED WIGNER OPERATOR AND QUANTIZATION SCHEME

Our aim is to construct s–parameterized quantization scheme, in another word, we want to construct a one-to-one correspondence between an operator and its classical correspondence in the sense of s–parameterized quasiprobability distribution. For this purpose we should introduce a generalized Wigner operator for the s–parameterized phase space theory. By analogy with the usual Wigner operator [6] we introduce a generalized Wigner operator for s–parameterized distributions,

$$\Delta_s(\alpha) = \int \frac{d^2\beta}{2\pi^2} \exp\left(\frac{s|\beta|^2}{2} + \beta a^\dagger - \beta^* a - \beta\alpha^* + \beta^*\alpha\right). \tag{10}$$

Using the Baker-Hausdorff formula to put the exponential in normally ordered form, and using the technique of integration within normal product of operators [10, 11], for $s < 1$, we obtain

$$\begin{aligned}\Delta_s(\alpha) &= \int \frac{d^2\beta}{2\pi^2} : \exp\left[\frac{-(1-s)|\beta|^2}{2} + \beta a^\dagger - \beta^* a - \beta\alpha^* + \beta^*\alpha\right] : \\ &= \frac{1}{(1-s)\pi} : \exp\left[\frac{-2}{1-s}\left(a^\dagger - \alpha^*\right)(a-\alpha)\right] : ,\end{aligned} \tag{11}$$

this is named s–parameterized Wigner operator. In particular, when $s = 0$, Eq. (11) reduces to the usual normally ordered Wigner operator [12]

$$\begin{aligned}\Delta_s(\alpha) \to \Delta(\alpha) &= \frac{1}{\pi} : \exp\left[-2\left(a^\dagger - \alpha^*\right)(a-\alpha)\right] : \\ &= \frac{1}{\pi} : \exp\left[-(x-X)^2 - (p-P)^2\right] : ,\end{aligned} \tag{12}$$

where $X = \frac{a^\dagger + a}{\sqrt{2}}, P = \frac{i\left(a^\dagger - a\right)}{\sqrt{2}}$. On the other hand, by putting the exponential in (10) within antinormal ordering symbol $\vdots\ \vdots$, we have for $s < -1$,

$$\begin{aligned}\Delta_s(\alpha) &= \int \frac{d^2\beta}{2\pi^2} \vdots \exp\left[\frac{-(-1-s)|\beta|^2}{2} + \beta a^\dagger - \beta^* a - \beta\alpha^* + \beta^*\alpha\right] \vdots \\ &= \frac{1}{(-1-s)\pi} \vdots \exp\left[\frac{2}{1+s}\left(a^\dagger - \alpha^*\right)(a-\alpha)\right] \vdots .\end{aligned} \tag{13}$$

In reference to the asymptotic expression of Delta function $\delta(x) = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{x^2}{\epsilon}}$, we have for $s = -1$,

$$\begin{aligned}\Delta_{s=-1}(\alpha) &= \vdots \delta\left(a^\dagger - \alpha^*\right)\delta(a-\alpha) \vdots = \delta(a-\alpha)\delta\left(a^\dagger - \alpha^*\right) \\ &= |\alpha\rangle\langle\alpha| ,\end{aligned} \tag{14}$$

which is the pure coherent state, $|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha a^\dagger\right)|0\rangle$. Using

$$: \exp\left[\left(e^\lambda - 1\right)a^\dagger a)\right] : = e^{\lambda a^\dagger a}, \tag{15}$$

we can convert (11) to the form

$$\begin{aligned}\Delta_s(\alpha) &= \frac{1}{(1-s)\pi} e^{\frac{2}{1-s}\alpha a^\dagger} : \exp\left[\left(\frac{s+1}{s-1} - 1\right)a^\dagger a\right] : e^{\frac{2}{1-s}\alpha^* a - \frac{2}{1-s}|\alpha|^2} \\ &= \frac{1}{(1-s)\pi} e^{\frac{2}{1-s}\alpha a^\dagger} e^{a^\dagger a \ln\frac{s+1}{s-1}} e^{\frac{2}{1-s}\alpha^* a - \frac{2}{1-s}|\alpha|^2}.\end{aligned} \tag{16}$$

III. THE s–PARAMETERIZED QUANTIZATION SCHEME

It follows from (11) that

$$2\int d^2\alpha\Delta_s\left(\alpha\right)=\frac{2}{\left(1-s\right)\pi}\int d^2\alpha\colon\ \exp\left[\frac{-2}{1-s}\left(a^{\dagger}-\alpha^{*}\right)\left(a-\alpha\right)\right]\colon\ =1, \tag{17}$$

so $\Delta_s\left(\alpha\right)$ is complete and ρ can be expanded as

$$\rho=2\int d^2\alpha\Delta_s\left(\alpha\right)\mathfrak{P}\left(\alpha\right), \tag{18}$$

which is a new classic-quantum mechanical correspondence between $\mathfrak{P}\left(\alpha\right)$ and ρ, when $s=0$, (18) yields the Weyl correspondence. By noting the form of $\Delta_{-s}\left(\alpha'\right)$ and using $\int\frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right|=1$ we calculate

$$\begin{aligned}
Tr\left[\Delta_{-s}\left(\alpha'\right)\Delta_s\left(\alpha\right)\right]&=G\int\frac{d^2z}{\pi}\left\langle z\right|e^{\frac{2}{1+s}\alpha' a^{\dagger}}e^{a^{\dagger}a\ln\frac{s-1}{s+1}}e^{\frac{2}{1+s}\alpha'^{*}a}e^{\frac{2}{1-s}\alpha a^{\dagger}}e^{a^{\dagger}a\ln\frac{s+1}{s-1}}e^{\frac{2}{1-s}\alpha^{*}a}\left|z\right\rangle\\
&=G\int\frac{d^2z}{\pi}\left\langle z\right|e^{\frac{2}{1+s}\alpha' a^{\dagger}}e^{\frac{-2}{1-s}\alpha'^{*}a}e^{\frac{-2}{1+s}\alpha a^{\dagger}}e^{\frac{2}{1-s}\alpha^{*}a}\left|z\right\rangle\\
&=Ge^{\frac{4\alpha'^{*}\alpha}{(1+s)(1-s)}}\int\frac{d^2z}{\pi}\exp[\frac{2z^{*}}{1+s}\left(\alpha'-\alpha\right)-\frac{2z}{1-s}\left(\alpha'^{*}-\alpha^{*}\right)]\\
&=\frac{1}{4\pi}\delta\left(\alpha'-\alpha\right)\left(\alpha'^{*}-\alpha^{*}\right)e^{-\left(\frac{2}{1-s}+\frac{2}{1+s}\right)|\alpha|^2-\frac{4\alpha'^{*}\alpha}{(1+s)(s-1)}}\\
&=\frac{1}{4\pi}\delta\left(\alpha'-\alpha\right)\left(\alpha'^{*}-\alpha^{*}\right)\\
&=\frac{1}{2\pi}\delta\left(q'-q\right)\left(p'-p\right),
\end{aligned} \tag{19}$$

where $G\equiv\frac{e^{-\frac{2}{1-s}|\alpha|^2-\frac{2}{1+s}|\alpha'|^2}}{(1+s)(1-s)\pi^2}$. Therefore, the classical function corresponding to ρ (in the context of the s–parameterized quantization scheme) is given by

$$\begin{aligned}
2\pi Tr\left[\Delta_{-s}\left(\alpha\right)\rho\right]&=4\pi\int d^2\alpha' Tr\left[\Delta_{-s}\left(\alpha\right)\Delta_s\left(\alpha'\right)\right]\mathfrak{P}\left(\alpha',s\right)\\
&=\int d^2\alpha'\delta\left(\alpha-\alpha'\right)\left(\alpha^{*}-\alpha'^{*}\right)\mathfrak{P}\left(\alpha',s\right)\\
&=\mathfrak{P}\left(\alpha,s\right).
\end{aligned} \tag{20}$$

Eq. (20) is the reciprocal relation of (18). Thus we have established one-to-one mapping between operators and their s–parameterized classical correspondence. The s–parameterized quantization scheme is completed, of which the Weyl quantization is its special case.

IV. EXPANSION FORMULA OF $\left|z\right\rangle\left\langle z\right|$ IN TERMS OF s–PARAMETERIZED QUANTIZATION SCHEME

When $\rho=\left|z\right\rangle\left\langle z\right|$, using (20) we have

$$\begin{aligned}
2\pi Tr\left[\Delta_{-s}\left(\alpha\right)\left|z\right\rangle\left\langle z\right|\right]&=\frac{2}{1+s}\left\langle z\right|\colon\ \exp\left[\frac{-2}{1+s}\left(a^{\dagger}-\alpha^{*}\right)\left(a-\alpha\right)\right]\colon\ \left|z\right\rangle\\
&=\frac{2}{1+s}\exp\left[\frac{-2}{1+s}\left(z^{*}-\alpha^{*}\right)\left(z-\alpha\right)\right],
\end{aligned} \tag{21}$$

this is the s-parameterized classical correspondence of $\left|z\right\rangle\left\langle z\right|$ in phase space. Eq. (21) represents a kind of phase space distribution, since the integration over it leads to the completeness

$$\int\frac{d^2z}{\pi}\left|z\right\rangle\left\langle z\right|\rightarrow\frac{2}{1+s}\int\frac{d^2z}{\pi}\exp\left[\frac{-2}{1+s}\left(z^{*}-\alpha^{*}\right)\left(z-\alpha\right)\right]=1. \tag{22}$$

For this s-parameterized distribution we can define s-ordered form of $|z\rangle\langle z|$ through the following formula

$$|z\rangle\langle z| = \frac{2}{1+s}\S\exp\left[\frac{-2}{1+s}\left(z^*-a^\dagger\right)(z-a)\right]\S, \tag{23}$$

where $\S\cdots\S$ means s-ordering symbol. This definition is consistent with those well-known ordered formulas of $|z\rangle\langle z|$. Indeed, when in (23) $s=0$, $\S\cdots\S$ converts to Weyl ordering $\vdots\cdots\vdots$, so (23) reduces to

$$|z\rangle\langle z| = 2\vdots\exp\left[-2\left(z^*-a^\dagger\right)(z-a)\right]\vdots, \tag{24}$$

as expected [7]; when in (23) $s=1$, $\S\cdots\S$ becomes normal ordering [10],

$$|z\rangle\langle z| =:\exp\left[-\left(z^*-a^\dagger\right)(z-a)\right]:, \tag{25}$$

which is as expected too; when $s=-1$, $\S\cdots\S$ becomes antinormal ordering,

$$\begin{aligned}|z\rangle\langle z| &= \lim_{\epsilon\to 0}\frac{2}{\epsilon}\vdots\exp\left[\frac{2}{\epsilon}\left(z^*-a^\dagger\right)(z-a)\right]\vdots \\ &= \vdots\delta\left(z^*-a^\dagger\right)\delta(z-a)\vdots,\end{aligned} \tag{26}$$

still is as expected.

V. THE TECHNIQUE OF INTEGRATION WITHIN s–ORDERED PRODUCT OF OPERATORS

Let us introduce the technique of integration within s-ordered product of operators (IWSOP) by listing some properties of the s-ordered product of operators which is defined through (23):

1. The order of Boson operators a and $a^\dagger$ within a s-ordered symbol can be permuted, even though $\left[a,a^\dagger\right]=1$.
2. c-numbers can be taken out of the symbol $\S\cdots\S$ as one wishes.
3. An s-ordered product of operators can be integrated or differentiated with respect to a c-number provided the integration is convergent.
4. The vacuum projection operator $|0\rangle\langle 0|$ has the s-ordered product form (see (23))

$$|0\rangle\langle 0| = \frac{2}{1+s}\S\exp\left(\frac{-2}{1+s}a^\dagger a\right)\S. \tag{27}$$

5. the symbol $\S\cdots\S$ becomes $:\,:$ for $s=1$, becomes $\vdots\,\vdots$ for $s=0$, and becomes $\vdots\,\vdots$ for $s=-1$.

VI. DENSITY OPERATOR'S EXPANSION FORMULA IN TERMS OF s–ORDERED QUANTIZATION SCHEME

Using (1) and (23) we have the expansion within $\S\cdots\S$,

$$\rho = \int\frac{d^2z}{\pi}P(z)|z\rangle\langle z| = \frac{2}{1+s}\int\frac{d^2z}{\pi}P(z)\S\exp\left[\frac{-2}{1+s}\left(z^*-a^\dagger\right)(z-a)\right]\S. \tag{28}$$

Substituting Mehta's expression of $P(z)$ [13]

$$P(z) = e^{|z|^2}\int\frac{d^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle e^{|\beta|^2+\beta^* z-\beta z^*}, \tag{29}$$

where $|\beta\rangle$ is also a coherent state, $\langle-\beta|\beta\rangle = e^{-2|\beta|^2}$, into (28) we have

$$\begin{aligned}\rho &= \tfrac{2}{1+s}\int\tfrac{d^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle e^{|\beta|^2}\int\tfrac{d^2z}{\pi}\S\exp\left[|z|^2+\beta^* z-\beta z^*-\tfrac{2}{1+s}\left(z^*-a^\dagger\right)(z-a)\right]\S \\ &= \tfrac{2}{1-s}\int\tfrac{d^2\beta}{\pi}\langle-\beta|\rho|\beta\rangle\S\exp\left[\tfrac{2}{s-1}\left(s|\beta|^2-\beta^* a+\beta a^\dagger-a^\dagger a\right)\right]\S,\end{aligned} \tag{30}$$

this is density operator's expansion formula in terms of s–ordered quantization scheme. In particular, when $s=0$, (30) becomes

$$\rho=2\int\frac{d^2\beta}{\pi}\left\langle -\beta\right|\rho\left|\beta\right\rangle \vdots\exp\left[2\left(\beta^{*}a-\beta a^{\dagger}+a^{\dagger}a\right)\right]\vdots, \tag{31}$$

which is the formula converting ρ into its Weyl ordered form [7, 8]; while for $s=-1$, (30) becomes

$$\rho=2\int\frac{d^2\beta}{\pi}\left\langle -\beta\right|\rho\left|\beta\right\rangle \vdots\exp\left[-\left(|\beta|^2+\beta^{*}a-\beta a^{\dagger}+a^{\dagger}a\right)\right]\vdots, \tag{32}$$

which is the formula converting ρ into its antinormally ordered form [14], as expected.

VII. APPLICATION

We now use (30) to derive the s–ordered expansion of $e^{\lambda a^{\dagger}a}$, using (15) and the IWSOP technique we have

$$\begin{aligned}
e^{\lambda a^{\dagger}a} &= \tfrac{2}{1-s}\int\tfrac{d^2\beta}{\pi}\left\langle -\beta\right|\exp\left[\left(1-e^{\lambda}\right)|\beta|^2\right]\left|\beta\right\rangle \S\exp\{\tfrac{2}{s-1}\left(s|\beta|^2-\beta^{*}a+\beta a^{\dagger}-a^{\dagger}a\right)\}\S \\
&= \tfrac{2}{1-s}\int\tfrac{d^2\beta}{\pi}\S\exp\left[\left(-1-e^{\lambda}\right)|\beta|^2+\tfrac{2}{s-1}\left(s|\beta|^2-\beta^{*}a+\beta a^{\dagger}-a^{\dagger}a\right)\right]\S \\
&= \tfrac{2}{1+s-se^{\lambda}+e^{\lambda}}\S\exp\left[\tfrac{2\left(e^{\lambda}-1\right)}{1+s-se^{\lambda}+e^{\lambda}}a^{\dagger}a\right]\S,
\end{aligned} \tag{33}$$

which is a new formula. For $s=1$, $\S\cdots\S\rightarrow:\ :$, (33) reduces to (15) as expected; for $s=0$, $\S\cdots\S\rightarrow\vdots\ \vdots$, (33) becomes the Weyl ordering expansion [7],

$$e^{\lambda a^{\dagger}a}=\frac{2}{1+e^{\lambda}}\vdots\exp\left[\frac{2e^{\lambda}-2}{1+e^{\lambda}}aa^{\dagger}\right]\vdots, \tag{34}$$

and for $s=-1$, $\S\cdots\S\rightarrow\vdots\ \vdots$, (33) becomes[10]

$$e^{\lambda a^{\dagger}a}=e^{-\lambda}\vdots\exp\left[\left(1-e^{-\lambda}\right)aa^{\dagger}\right]\vdots, \tag{35}$$

which is also correct. Further, we consider the s–ordered expansion of the generalized Wigner operator itself, using (10) and (30) we have

$$\begin{aligned}
\Delta_s\left(\alpha^{*},\alpha\right) &= \tfrac{2}{(1-s)^2\pi}\int\tfrac{d^2\beta}{\pi}\left\langle -\beta\right|:\ \exp\left[\tfrac{-2}{1-s}\left(a^{\dagger}-\alpha^{*}\right)\left(a-\alpha\right)\right]:\ \left|\beta\right\rangle \\
&\quad\times\S\exp\left[\tfrac{2}{s-1}\left(s|\beta|^2-\beta^{*}a+\beta a^{\dagger}-a^{\dagger}a\right)\right]\S \\
&= \tfrac{2}{(1-s)^2\pi}\int\tfrac{d^2\beta}{\pi}\S e^{-2|\beta|^2}\exp\left\{\tfrac{2}{s-1}\left[\left(-\beta^{*}-\alpha^{*}\right)\left(\beta-\alpha\right)\right.\right. \\
&\quad\left.\left.+s|\beta|^2-\beta^{*}a+\beta a^{\dagger}-a^{\dagger}a\right]\right\}\S \\
&= \tfrac{2}{(1-s)^2}\S\delta\left[\tfrac{2}{s-1}\left(a^{\dagger}-\alpha^{*}\right)\right]\delta\left[\tfrac{2}{s-1}\left(a-\alpha\right)\right]\S,
\end{aligned} \tag{36}$$

which in the case of $s=0$ becomes the Weyl ordered form of the usual Wigner operator $\Delta\left(\alpha\right)=\frac{1}{2}\vdots\delta\left(\alpha^{*}-a^{\dagger}\right)\delta\left(\alpha-a\right)\vdots$.

In summary, by introducing the s–parameterized generalized Wigner operator into phase-space quantum mechanics we have proposed the technique of integration within s–ordered product of operators (which considers normal ordered, antinormally ordered and Weyl ordered product of operators as its special cases). The s–ordered operator expansion (denoted by $\S\cdots\S$) formula of density operators is derived. The theory of Integration within normal-, Weyl- and antinormal-ordering of operators can now be tackled in a unified way. The s–parameterized quantization scheme is completely established, of which the Weyl quantization is its special case. For the mutual transformation between the Weyl ordering and $X-P$ (or $P-X$) ordering of operators we refer to [15].

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